Math for Management, Winter 2023

# List 3

*Matrices, systems of linear equations*

A **matrix** is a grid of numbers. The **dimensions** of a matrix are written in the format "$m \times n$", spoken as "$m$ by $n$", where $m$ is the number of rows and $n$ is the number of columns (write both numbers; do not multiply them).

43. Give the dimensions of the following matrices:

(a) $\begin{bmatrix} -92 & 8 \\ -78 & -67 \end{bmatrix}$

(b) $\begin{bmatrix} -36 \\ 72 \\ -12 \end{bmatrix}$

(c) $\begin{bmatrix} 75 & 89 & 50 \\ -5 & -81 & 34 \end{bmatrix}$

(d) $\begin{bmatrix} -13 & -63 & -5 \\ 0 & -66 & \frac{1}{2} \\ 31 & \frac{5}{22} & \frac{8}{11} \end{bmatrix}$

(e) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(f) $\begin{bmatrix} 58 & -65 & 40 & 8 & -1 & 26 \\ -74 & 58 & -92 & -4 & -21 & 74 \end{bmatrix}$

In order to for the matrix product $MN$ to exist (that is, for it to be possible to do the multiplication $MN$) it must be that the number of columns of $M$ is equal to the number of rows of $N$.

44. If $A$ is a $2 \times 2$ matrix, $B$ is a $3 \times 3$ matrix, and $C$ is a $3 \times 2$ matrix, which of the following exist?

| | | | |
|---|---|---|---|
| (a) $AA$ | (e) $BB$ | (i) $CC$ | (m) $A^\top C$ |
| (b) $AB$ | (f) $BC$ | (j) $ABC$ | (n) $AC^\top$ |
| (c) $AC$ | (g) $CA$ | (k) $BCA$ | (o) $C^\top C$ |
| (d) $BA$ | (h) $CB$ | ($\ell$) $ACA$ | (p) $AB^\top CAC^\top$ |

45. (a) Calculate $\begin{bmatrix} 1 & 2 \\ 5 & -8 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 2 & 12 \end{bmatrix}$. (b) Calculate $\begin{bmatrix} 3 & 0 \\ 2 & 12 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 5 & -8 \end{bmatrix}$.

(b) Compare your answers to parts (a) and (b).

The **transpose** of a matrix $M$, written $M^\top$, swaps the rows and columns.
For example, $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^\top = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$.

46. Compute the following:

(a) $\begin{bmatrix} 1 & 14 & 8 \\ 6 & -1 & 14 \\ 5 & 11 & -3 \end{bmatrix} + \begin{bmatrix} 11 & 2 & 10 \\ -2 & 14 & 8 \\ -2 & 3 & -4 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 14 & 8 \\ 6 & -1 & 14 \\ 5 & 11 & -3 \end{bmatrix} - \begin{bmatrix} 11 & 2 & 10 \\ -2 & 14 & 8 \\ -2 & 3 & -4 \end{bmatrix}$

(c) $3\begin{bmatrix} 0 & -4 & 0 \\ -1 & -1 & 3 \\ -2 & 5 & 14 \end{bmatrix}$

(d) $\frac{1}{6}\begin{bmatrix} 9 & 14 \\ 6 & 10 \end{bmatrix}$

(e) $\begin{bmatrix} 8 & 5 \\ 0 & -5 \end{bmatrix}\begin{bmatrix} 1 \\ 5 \end{bmatrix}$

(f) $\begin{bmatrix} 9 & 8 \\ -2 & 5 \end{bmatrix}^\top \begin{bmatrix} 2 \\ 6 \end{bmatrix}$

(g) $\begin{bmatrix} -5 & 5 & 7 \\ -2 & -3 & 2 \\ -2 & 1 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 9 \\ 8 \end{bmatrix}$

(h) $\begin{bmatrix} 4 & 8 & 0 \\ -3 & 3 & -3 \\ 8 & 5 & -2 \end{bmatrix} \begin{bmatrix} -2 \\ 4 \\ 7 \end{bmatrix}$

(i) $\begin{bmatrix} 4 & 8 & 0 \\ -3 & 3 & -3 \\ 8 & 5 & -2 \end{bmatrix}^\top \begin{bmatrix} -2 \\ 4 \\ 7 \end{bmatrix}$

47. Compute $\begin{bmatrix} 1 & -\sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & \sqrt{2} & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix}$.

48. Compute the following, if they exist:

(a) $\begin{bmatrix} 9 & -4 \\ -5 & -5 \end{bmatrix} \begin{bmatrix} 8 & 1 \\ 0 & -3 \end{bmatrix}$

(b) $\begin{bmatrix} 4 & 5 & 22 \\ 8 & -13 & 4 \end{bmatrix} \begin{bmatrix} 19 & 0 & 35 & 6 \\ 0 & 2 & 2 & 6 \\ 9 & 1 & 19 & -1 \end{bmatrix}$

(c) $\begin{bmatrix} 19 & 0 & 35 & 6 \\ 0 & 2 & 2 & 6 \\ 9 & 1 & 19 & -1 \end{bmatrix} \begin{bmatrix} 4 & 5 & 22 \\ 8 & -13 & 4 \end{bmatrix}$

(d) $\begin{bmatrix} 3 & 0 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 0 & 2 & -1 & 2 & 7 \\ 3 & -4 & -1 & 1 & 8 \end{bmatrix}$

(e) $\begin{bmatrix} -2 & -4 \\ 7 & 5 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 7 & 8 \\ 2 & 8 \end{bmatrix}$

(f) $\begin{bmatrix} -4 & -3 & -5 \\ 24 & 6 & 29 \end{bmatrix} \begin{bmatrix} 4 & 13 & 0 \\ 2 & -26 & 9 \end{bmatrix}$

(g) $\begin{bmatrix} -4 & -3 & -5 \\ 24 & 6 & 29 \end{bmatrix} \begin{bmatrix} 4 & 13 & 0 \\ 2 & -26 & 9 \end{bmatrix}^\top$

49. (a) Calculate $\left( \begin{bmatrix} 1 & 2 \\ 5 & -8 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 2 & 12 \end{bmatrix} \right)^\top$.

(b) Calculate $\begin{bmatrix} 1 & 2 \\ 5 & -8 \end{bmatrix}^\top \begin{bmatrix} 3 & 0 \\ 2 & 12 \end{bmatrix}^\top$.

(c) Calculate $\begin{bmatrix} 3 & 0 \\ 2 & 12 \end{bmatrix}^\top \begin{bmatrix} 1 & 2 \\ 5 & -8 \end{bmatrix}^\top$.

(d) Compare your answers to parts (a) and (b).

(e) Compare your answers to parts (a) and (c).

50. Compute the following:

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 8 & 2 \\ 3 & -3 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 14 & 21 \\ -11 & 23 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 99 & \frac{1}{10} \\ -37 & 2 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -4 & -2 & 1 \\ 5 & 2 & 5 \\ 7 & 4 & 1 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 15 & 11 & -15 \\ 17 & 10 & -8 \\ 0 & 0 & -13 \end{bmatrix}$

(f) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 59 & 28 & 58 \\ 61 & 44 & 67 \\ 22 & 39 & 17 \end{bmatrix}$

(g) $\begin{bmatrix} 59 & 28 & 58 \\ 61 & 44 & 67 \\ 22 & 39 & 17 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

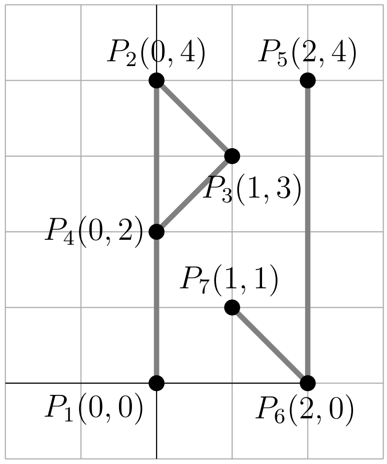
(h) $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -39 & -66 & 84 & 66 & -10 \\ -47 & -5 & 17 & -59 & -3 \\ -94 & -90 & -5 & 86 & 31 \\ 25 & 80 & 0 & 35 & 19 \\ -72 & 40 & 99 & 48 & 57 \end{bmatrix}$

51. For each of the points $P_1$ through $P_7$, calculate

$$P_i{}' = \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix} P_i.$$

(For example, for $P_5{}' = \left[\begin{smallmatrix} 1 & 1/2 \\ 0 & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} 2 \\ 4 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 4 \\ 4 \end{smallmatrix}\right]$.) Plot the points $P_1{}', \ldots, P_7{}'$ on a new grid. Connect $P_1{}' \to P_2{}' \to P_3{}' \to P_4{}'$ with line segments, and connect $P_5{}' \to P_6{}' \to P_7{}'$.

*Congratulations. You can write in italics!*




52. If $\begin{bmatrix} 3 & 5 \\ 5 & 9 \end{bmatrix} M = \begin{bmatrix} 8 & 25 & 12 \\ 14 & 45 & 22 \end{bmatrix}$, what are the dimensions of matrix $M$?

53. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}$, $C = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $D = \begin{bmatrix} 0 & 5 & 2 \end{bmatrix}$, and $E = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 3 & 1 \end{bmatrix}$. Write all the products of two matrices from this list that exist (e.g., $AA$ exists, but $AC$ does not).

54. For each of the following equations, either give the dimensions of the matrix $M$ or state that such a matrix does not exist. (You do *not* have to solve for $M$.)

(a) $M = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$

(b) $M = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}$

(c) $M = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

(d) $M = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

(e) $M = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

(f) $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} M \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

(g) $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} M \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(h) $\begin{bmatrix} 2 & -8 \\ 1 & 5 \\ 0 & -7 \end{bmatrix} \begin{bmatrix} 9 & 0 & 0 & 11 & 4 \\ -2 & -8 & 6 & 1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 5 \\ -4 \\ 0 \\ 1 \\ -9 \end{bmatrix} \begin{bmatrix} \frac{2}{7} & 1 & \frac{4}{7} \end{bmatrix} = M$

Earlier versions of Tasks 55 and 56 involved "determinants", which are not part of MAT 1448.

55. Suppose $M$ is a $5 \times 12$ matrix. Can there be a matrix $N$ such that both $MN$ and $NM$ exist? If so, can anything be said about the dimensions of $N$?

56. Calculate $\begin{bmatrix} 11 & \frac{9}{2} \\ -2 & 21 \end{bmatrix}^2$ and $\begin{bmatrix} -16 & 18 \\ -8 & 24 \end{bmatrix}^2$ and compare the answers.

The $\boldsymbol{n \times n}$ **identity matrix** is the matrix $I$ (also written $I_n$ or $I_{n\times n}$) such that

$$IM = MI = M$$

for any $n \times n$ matrix $M$. It has 1 along the main diagonal and 0 everywhere else.

57. (a) Multiply $\begin{bmatrix} 0 & -4 & 2 \\ 10 & -1 & 0 \\ 1 & 12 & -6 \end{bmatrix} \begin{bmatrix} 3 & 0 & 1 \\ 30 & -1 & 10 \\ \frac{121}{2} & -2 & 20 \end{bmatrix}$.

(b) Multiply $\begin{bmatrix} 0 & -4 & 2 \\ 10 & -1 & 0 \\ 1 & 12 & -6 \end{bmatrix} \begin{bmatrix} 3 & 0 & 1 \\ 30 & -1 & 10 \\ \frac{121}{2} & -2 & 20 \end{bmatrix} \begin{bmatrix} 6 & 4 & 9 \\ 0 & -3 & -2 \\ 4 & 6 & -4 \end{bmatrix}$.

The **inverse matrix** of a matrix $M$ is written $M^{-1}$ (spoken as "M inverse") and it is the unique matrix for which $M^{-1}\,M = I$ and $M\,M^{-1} = I$. For a $2\times 2$ matrix,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}.$$

For any square matrix, the inverse can be found by carefully applying "row operations" to the "augmented matrix" $[\,M \,|\, I\,]$ until it becomes $[\,I \,|\, M^{-1}]$.

58. Find $\begin{bmatrix} 5 & 4 \\ 1 & -2 \end{bmatrix}^{-1}$ and $\begin{bmatrix} 0 & -4 & 2 \\ 10 & -1 & 0 \\ 1 & 12 & -6 \end{bmatrix}^{-1}$.

59. Find the inverses of the following matrices. Compare the answers of parts (e) and (f).

(a) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1}$

(b) $\begin{bmatrix} 3 & 5 \\ 5 & 9 \end{bmatrix}^{-1}$

(c) $\begin{bmatrix} 8 & 4 \\ 6 & 3 \end{bmatrix}^{-1}$

(d) $\begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix}^{-1}$

(e) $\begin{bmatrix} 3 & b \\ 2 & 5 \end{bmatrix}^{-1}$

(f) $\begin{bmatrix} 3 & 2 \\ b & 5 \end{bmatrix}^{-1}$

(g) $\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 9 \\ 5 & 6 & 8 \end{bmatrix}^{-1}$

☆(h) $\begin{bmatrix} 3 & 1 & 3 & 3 \\ 1 & 2 & 4 & 0 \\ 2 & 0 & 2 & 2 \\ 4 & 2 & 1 & 3 \end{bmatrix}^{-1}$

60. Find the matrix $M$ from Task 52.

61. Solve the following matrix equations:

(a) $X\begin{bmatrix} -1 & 1 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 3 & 4 \end{bmatrix}$.

(b) $\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} X \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 2 & 2 \end{bmatrix}$.

(c) $\left(\begin{bmatrix} 0 & 3 \\ 5 & -2 \end{bmatrix} + 4X\right)^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

(d) $3\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix} = X^{\top}\begin{bmatrix} 5 & 7 \\ 6 & 8 \end{bmatrix}$.

Earlier versions of Tasks 62 and parts of Task 63 also involved determinants.

62. Find a matrix $X$ for which $\begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix} X = \begin{bmatrix} 10 & 4 \\ 10 & 4 \end{bmatrix}$.

63. For each of the following, does an inverse matrix exist?

(a) the $3 \times 3$ identity matrix.

(b) a $3 \times 5$ matrix where every number in the matrix is 1.

(c) a $4 \times 4$ matrix where every number in the matrix is 1.

(d) a $4 \times 4$ matrix where every number in the matrix is 0.

(e) a $2 \times 2$ matrix with $a_{ij} = i + j$.

64. Use inverse matrices to solve these systems:

(a) $2x - y = 3,\ 3x + y = 2$

(b) $x + 2y = 0,\ 2x - y = 5$

(c) $\begin{cases} x + y + z = 5 \\ 2x + 2y + z = 3 \\ 3x + 2y + z = 1 \end{cases}$

(d) $\begin{cases} x + y + z = 4 \\ 2x - 3y + 5z = -5 \\ -x + 2y - z = 2 \end{cases}$

65. Use the Gauss method to solve the systems of linear equations from Task 64.

66. Solve the following systems of equations:

(a) $\begin{cases} x + 2y + 3z = 14 \\ 4x + 3y - z = 7 \\ x - y + z = 2 \end{cases}$

(b) $\begin{cases} 3x + 4y + z + 2t = 3 \\ 6x + 8y + 2z + 5t = 7 \\ 9x + 12y + 3z + 10t = 13 \end{cases}$

(c) $\begin{cases} 3x - 5y + 2z + 4t = 2 \\ 7x - 4y + z + 3t = 5 \\ 5x + 7y - 4z - 6t = 3 \end{cases}$

(e) $\begin{cases} 3x + 2y + 2z + 2t = 2 \\ 2x + 3y + 2z + 5t = 3 \\ 9x + y + 4z - 5t = 1 \\ 2x + 2y + 3z + 4t = 5 \\ 7x + y + 6z - t = 7 \end{cases}$